Technical Comments

Direct Blunt-Body Integral Method

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Introduction

GENERAL remarks by Dorodnitsyn[1] point out that an approximate solution obtained by using the method of integral relations depends, among other things, on the choice of the coordinate system. Because of simplicity, body-oriented coordinates are usually preferred for the supersonic blunt-body problem, and considerable results have been published.[2-4] Unfortunately, there are a number of applications where nonbody-oriented coordinates are necessary, as the choice is dictated by considerations other than the simplicity of the blunt-body formulation. Some results using nonbody-oriented coordinates for simple axisymmetric bodies have also been presented.[5,6] Recent experience[7] with an application of the method in the latter category suggests that a few comments about some of its more subtle characteristics might be helpful.

Analysis

As a specific example, let us consider planar flow about arbitrary symmetric bodies described in polar coordinates (r, θ). The limiting case of the circular cylinder then corresponds to body-oriented coordinates. Usually the governing equations which are chosen are the momentum equation in the r direction

$$(\partial s/\partial\theta) + [\partial (rH)/\partial r] = g \tag{1}$$

and the continuity equation

$$(\partial t/\partial\theta) + [\partial(rh)/\partial r] = 0 \tag{2}$$

supplemented by the Bernoulli integral $p = \rho(1 - w^2)$ and conservation of entropy $\Phi(\psi) = p/\rho^\gamma$. The nomenclature and nondimensionalization used here corresponds to that used by Belotserkovskii.[2] The first approximation of Belotserkovskii, where s, t, and g are approximated along the radius vector by a linear function of distance, then reduces Eqs. (1) and (2) after integration in the r direction to the ordinary differential equations

$$s_0' = A - s_1' \tag{3}$$

and

$$t_0' = B - t_1' \tag{4}$$

where the subscripts 0 and 1 refer to values on the body and at the shock, respectively. The prime denotes total differentiation with respect to θ. From $s_0 = s_0(u_0, v_0)$ and $t_0 = t_0(u_0, v_0)$ one writes

$$s_0' = (\partial s_0/\partial u_0)u_0' + (\partial s_0/\partial v_0)v_0' \tag{5}$$

and

$$t_0' = (\partial t_0/\partial u_0)u_0' + (\partial t_0/\partial v_0)v_0' \tag{6}$$

In addition, the shock wave relations yield

$$s_1' = (\partial s_1/\partial\sigma)\sigma' + (\partial s_1/\partial\theta) \tag{7}$$

$$t_1' = (\partial t_1/\partial\sigma)\sigma' + (\partial t_1/\partial\theta) \tag{8}$$

where σ is the inclination of the shock wave to the direction of the incident stream. The terms A and B, and the partial derivatives in Eqs. (5-8) are given functionally elsewhere.[7] When Eqs. (3-8) are supplemented by the relationship between u_0' and v_0'

$$u_0' = \eta v_0' + v_0\eta' \tag{9}$$

where

$$\eta = r_0'/r_0 \tag{10}$$

and η' describes the body contour, one can solve for the total derivatives in terms of the partial derivatives. The equations for v_0' and σ' so obtained are

$$v_0' = E/D \tag{11}$$

and

$$\sigma' = \alpha\, v_0' + \beta \tag{12}$$

Equations (11) and (12) along with

$$r_1' = -r_1 \cot(\sigma + \theta) \tag{13}$$

obtained from geometric considerations, lead to the usual set of three ordinary differential equations, which must be integrated simultaneously to define the shock wave and the body flow properties in the first approximation. If η and η' are not given, but are obtained as part of the solution with other governing equations, then Eqs. (9) and (10) may be integrated for u_0 and r_0 simultaneously with Eqs. (11-13). Such an application arises when the body consists of a fluid in motion, and r_0 gives the position of the dividing streamline across which the static pressure remains continuous.[7]

Discussion

In terms of known or calculable functions, D, E, α, and β are given by

$$D = \frac{\partial s_1}{\partial\sigma}\left(\frac{\partial t_0}{\partial v_0} + \eta\,\frac{\partial t_0}{\partial u_0}\right) - \frac{\partial t_1}{\partial\sigma}\left(\frac{\partial s_0}{\partial v_0} + \eta\,\frac{\partial s_0}{\partial u_0}\right) \tag{14}$$

$$E = \frac{\partial s_1}{\partial\sigma}\left(B - \frac{\partial t_1}{\partial\theta} - v_0\,\eta'\,\frac{\partial t_0}{\partial u_0}\right) - \frac{\partial t_1}{\partial\sigma}\left(A - \frac{\partial s_1}{\partial\theta} - v_0\,\eta'\,\frac{\partial s_0}{\partial u_0}\right) \tag{15}$$

$$\alpha = -[(\partial s_0/\partial v_0) + \eta(\partial s_0/\partial u_0)]/(\partial s_1/\partial\sigma) \tag{16}$$

and

$$\beta = [A - (\partial s_1/\partial\theta) - v_0\,\eta'(\partial s_0/\partial u_0)]/(\partial s_1/\partial\sigma) \tag{17}$$

The governing equations are usually integrated numerically

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starting at the axis of symmetry where the shock-layer thickness $\epsilon = r_1 - r_0$ is unknown. The stand-off distance $\epsilon(0)$ is determined through an iteration procedure by the requirement that the regularity condition for Eq. (11), $E = 0$ when $D = 0$, must be satisfied.

For the circular cylinder $s_0 = 0$, the floating singularity occurs when the product $(1 - M_0^2)\partial s_1/\partial\sigma$ is zero. Obviously the body sonic point is a point of singularity as in all of the body-oriented systems. Accurate location and smooth passage through the singular point is generally very troublesome since integration is unstable in its vicinity. Kao[8] has recently shown that in a body-oriented coordinate system one can shift the singularity from the differential equations to an algebraic expression by a dependent variable transformation, i.e., $v_0 \rightarrow t_0$. Such a transformation changes the nature of the difficulty, but does not completely eliminate it. Furthermore, if the coordinate system is not body-oriented, then the structure of the system of equations is such that this shift cannot be accomplished.

One also observes in the circular cylinder case that, if $\partial s_1/\partial\sigma$ should pass through zero, then generally a physically meaningful solution no longer exists. The exception occurs when $\partial s_1/\partial\sigma$ passes through zero at the sonic point. In calculations of the flow past cylindrical bodies, where the usual first approximation is apparently poor, $\partial s_1/\partial\sigma$ does pass through zero following the sonic point in most cases. The fortunate circumstance, when $\partial s_1/\partial\sigma$ is zero near enough to the sonic point to allow smooth passage through the singular region, occurs for a cylinder at a nominal Mach number of 4 when $\gamma = 1.4$. The confusing difficulty mentioned has been encountered previously, as it was reported in Ref. 4 that the relatively simple convergence procedure developed for axisymmetric bodies was never satisfactory for the circular cylinder. Xerikos further speculated that the additional singular behavior of σ' might indicate that the first approximation is much poorer in the two-dimensional problem than for the axisymmetric problem.

The nature of the saddle-point singularity is somewhat more complex for the noncircular case. In the more general case, the singularity occurs when the local Mach number at the body attains the value $(1 - 2\delta)^{1/2}/(1 - \delta)^{1/2}$ where $\delta = u_0 \Phi_0^{1/1-\gamma} (\partial t_1/\partial\sigma)/(\partial s_1/\partial\sigma)$. This result has not been clearly stated in the literature, although the equations in a nonbody-oriented coordinate system have been correctly presented in Ref. 5. An incorrect formulation appears in Ref. 6 where it is stated that the body singularity is located at the sonic point for arbitrary axisymmetric bodies treated using polar coordinates, when in fact it is true only for the sphere. The apparent agreement of the results in Ref. 6 with experiments and with Ref. 5 appears to be due to the origin of the coordinates being taken at the nose center of curvature which located the singularity near the sonic point. Obviously, if the radius vector is normal to the body surface at the singular point, then it is also the sonic point. The physical significance of the line of singularities present for higher approximations is discussed in Ref. 9. In nonbody-oriented coordinates, the situation is complicated at the body since additional information then enters the problem.

The sign of δ controls whether the body singularity is located in the subsonic or the supersonic region. The fact that the magnitude and sign of δ depend on variables at the shock wave and body surface immediately give rise to the possibility that D may possess more than one zero. This condition may be caused by the nature of the body contour or through the behavior of $\partial s_1/\partial\sigma$ and $\partial t_1/\partial\sigma$, which is analogous to the manner in which a second singularity can occur in body-oriented coordinates. In cases where a second singularity occurs in the supersonic region, the method of characteristics can be used to continue the solution. In any event, a severe limitation is introduced in supersonic regions, since body information is transmitted to the field along a coordinate direction, rather than along a characteristic direction. If a second singularity occurs in the subsonic region, then the only recourse appears to be the use of better approximations.

There are a number of ways to obtain better results aside from changing the basic coordinate system. One must attempt to get the best results obtainable from a low approximation, as the method soon becomes impractical when higher approximations are required. Where it is possible, a simple translation of the origin of the coordinates may be the necessary change. In other cases, the approximation can be modified through the use of suitable weighting functions or different interpolation formulas. The results of a given approximation also can be improved through the choice of different governing equations into which the approximations are introduced, since different functions are then approximated. If higher degree polynomials are required to approximate the integrands, it seems preferable to use data on the strip boundaries to evaluate the coefficients as was proposed in Ref. 3. Such a method is more in the Karman-Pohlhausen spirit, and additional free parameters will not be present as when additional strips are introduced.

It can be seen from the preceding discussion that more fundamental information is needed about the extent to which a given system of partial differential equations can be represented by a system of ordinary differential equations obtained through the method of integral relations. In any case, lack of convergence of the approximate blunt-body solution to the exact solution is indicated by the existence of a "fictitious transonic frontier" even as the degree of the approximating polynomials tends to infinity.[10]

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